# proof of Cramer's rule*

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Since $\det(A) \neq 0$, by properties of the determinant we know that $A$ is invertible.

We claim that this implies that the equation $Ax = b$ has a unique solution. Note that $A^{-1}b$ is a solution since $A(A^{-1}b) = (AA^{-1})b = b$, so we know that a solution exists.

Let $s$ be an arbitrary solution to the equation, so $As = b$. But then $s = (A^{-1}A)s = A^{-1}(As) = A^{-1}b$, so we see that $A^{-1}b$ is the only solution.

For each integer $i$, $1 \leq i \leq n$, let $a_i$ denote the $i$th column of $A$, let $e_i$ denote the $i$th column of the identity matrix $I_n$, and let $X_i$ denote the matrix obtained from $I_n$ by replacing column $i$ with the column vector $x$.

We know that for any matrices $A, B$ that the $k$th column of the product $AB$ is simply the product of $A$ and the $k$th column of $B$. Also observe that $Ae_k = a_k$ for $k = 1, \ldots, n$. Thus, by multiplication, we have:

$$\begin{aligned} AX_i &= A(e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n) \\ &= (Ae_1, \ldots, Ae_{i-1}, Ax, Ae_{i+1}, \ldots, Ae_n) \\ &= (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \\ &= M_i \end{aligned}$$

Since $X_i$ is $I_n$ with column $i$ replaced with $x$, computing the determinant of $X_i$ with cofactor expansion gives:

$$\det(X_i) = (-1)^{(i+i)} x_i \det(I_{n-1}) = 1 \cdot x_i \cdot 1 = x_i$$

Thus by the multiplicative property of the determinant,

$$\det(M_i) = \det(AX_i) = \det(A)\det(X_i) = \det(A)x_i$$

and so $x_i = \frac{\det(M_i)}{\det(A)}$ as required.

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